Contour Integration

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In *Surely You're Joking, Mr. Feynman!*, Richard Feynman wrote:

> One time I boasted, "I can do by other methods any integral anybody else needs contour integration to do." So Paul [Olum] puts up this tremendous damn integral he had obtained by starting out with a complex function that he knew the answer to, taking out the real part of it and leaving only the complex part. He had unwrapped it so it was only possible by contour integration! He was always deflating me like that. He was a very smart fellow.

In this section, we will make the utility of contour integration concrete.

1 Analytic Functions

In my experience, it's best to put the definitions up front, so that when we begin working with them later everything falls into place more naturally. Being clear at the outset about the kinds of functions we are dealing with makes it easier to see how the pieces connect, and having the terminology in hand will help the later results tie together.

A function $f : U \to \mathbb{C}$ on an open set $U \subset \mathbb{C}$ can be classified in increasingly restrictive ways:

1. **Complex differentiable (at a point).** A function is differentiable at $z_0 \in U$ if the limit

   $$f'(z_0) \ = \ \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

   exists. Unlike the real case, the approach $z \to z_0$ can come from any direction in the complex plane. This makes complex differentiability much more restrictive. Example: $f(z) = |z|^2$ is real-differentiable everywhere, but not complex-differentiable anywhere except at $z = 0$.

2. **Holomorphic.** A function is holomorphic on an open set U if it is complex-differentiable at every point of U. Equivalently: f satisfies the *Cauchy–Riemann equations* (relations among partial derivatives which ensure that the complex derivative is the same no matter from which direction z approaches z_0; see §2.4), and the partial derivatives of f are continuous. Another fact is that holomorphic functions are infinitely differentiable (in both the real and complex sense). However, the converse is not true: a function can be infinitely differentiable (C^∞) as a real function without being holomorphic. Functions such as $f(z) = e^z,\ f(z) = z^n,\ f(z) = \sin z$ are holomorphic everywhere.

3. **Analytic.** A function is analytic at $z_0 \in U$ if, in some neighborhood of z_0, it can be expressed as a convergent power series:

   $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

   In real analysis, analyticity is stronger than being C^∞, since there exist smooth functions that are not analytic. In complex analysis, however, one has the fundamental equivalence

   $$\text{holomorphic on } U \quad \Longleftrightarrow \quad \text{analytic on } U.$$

We'll see in §2.5 why this is true. For now, consider the ODE $f'(z) = f(z)$ with $f(0) = 1$. Requiring f to be analytic pins down a unique solution: $f(z) = e^z$. Once you know the rule it satisfies and its value at a single point, the function everywhere in the domain is determined. This is what makes analytic functions so powerful: local information dictates global behavior.

The domain U is assumed to be open (i.e., it doesn't include its boundary) because differentiability and analyticity are *local* properties: to check whether f is holomorphic at z_0, we need values of $f(z)$ for z in a neighborhood of z_0. If U were closed, there would be no room to approach a point z_0 on the boundary from all complex directions, and the derivative might not be well-defined.

1.1 Complex Logarithms

We can use complex numbers to analytically continue otherwise inaccessible regions of common functions. Every nonzero complex number can be written in polar form

$$z = r\,e^{i\theta}, \qquad r > 0,\ \theta \in \mathbb{R}.$$

The *argument* θ is only defined up to adding multiples of 2π: turning once around the origin does not change z but it does change the angle by 2π.

A (complex) logarithm of z is any w satisfying $e^w = z$. Write $w = a + ib$ with $a, b \in \mathbb{R}$. Then

$$e^w = e^{a+ib} = e^a e^{ib} = e^a\big(\cos b + i \sin b\big).$$

Matching magnitude and angle with $z = re^{i\theta}$ gives

$$e^a = r \quad \Rightarrow \quad a = \ln r, \qquad b \equiv \theta \pmod{2\pi} \quad \Rightarrow \quad b = \theta + 2\pi k,\ k \in \mathbb{Z}.$$

Hence

$$w = \ln r + i(\theta + 2\pi k).$$

Which we can verify directly:

$$e^{\ln r + i(\theta + 2\pi k)} = e^{\ln r}\,e^{i\theta}\,e^{i2\pi k} = r\,e^{i\theta} \cdot 1 = z.$$

Therefore the *multivalued* logarithm is

$$\log z = \ln r + i(\theta + 2\pi k), \qquad k \in \mathbb{Z}.$$

Writing $z = re^{i\theta}$, the logarithm "wants" to be $\ln r + i\theta$, but θ is only defined modulo 2π. If you walk once around 0, $\theta \mapsto \theta + 2\pi$ and $\ln r + i\theta$ jumps by $2\pi i$. Picture a *spiral staircase*: each full turn takes you to the next floor a height $2\pi i$ above. A *branch* means choosing a single floor by fixing a continuous range for θ.

To make $\log z$ single-valued, we *choose* a continuous range for the angle and cut the plane along the line where the angle would jump. The standard choice (the principal branch) is

$$\operatorname{Arg} z \in (-\pi, \pi], \qquad \operatorname{Log} z \ \equiv\ \ln|z| + i\ \operatorname{Arg} z,$$

with a straight *branch cut* along the negative real axis (note the distinction in notation: i.e. Log vs. log when referring to the principal branch). The main reason we cut there is so that the positive real axis is included smoothly, with no jump. That is, on the positive real axis ($z = x > 0$): the angle is $\theta = 0$, so $\operatorname{Log} x = \ln x$, which matches the ordinary real logarithm. If we had instead chosen $\theta \in (0, 2\pi)$, the "jump" would be at $\theta = 0 = 2\pi$, i.e. along the positive real axis. That would make $\operatorname{Log} x$ discontinuous for positive x, which is very undesirable.

On that choice,

$$\mathrm{Log}(-1) = \ln 1 + i\,\pi = i\pi.$$

Approaching -1 from *below* the cut gives angle $-\pi$, so the limiting value there is $-\,i\pi$. The jump across the cut is therefore

$$\mathrm{Log}\left((-1)^{+}\right) - \mathrm{Log}\left((-1)^{-}\right) = i\pi - (-i\pi) = 2\pi i.$$

This $2\pi i$ jump is the hallmark of the logarithm's multivalued nature.

2 Integration Along an Arc

When we first meet integrals, we picture them as areas under curves. That works in one dimension, but in the complex plane there is no natural "under." Here $f(z)$ is itself a complex number, and a complex number can be thought of as an arrow in the plane: it has a length $|f(z)|$ and a direction $\arg f(z)$.

Now think of a contour γ as a path you walk along. At each instant your motion has a "velocity vector", tangent to the curve, given by $dz = z'(t)\,dt$. This dz is the tiny arrow pointing exactly in the direction you are walking.

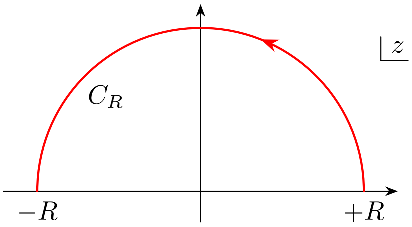
The function value $f(z)$ then acts on this arrow: it stretches it by $|f(z)|$ and rotates it by $\arg f(z)$. So each contribution to the integral is a transformed step, and the contour integral

$$\int_{\gamma} f(z)\,dz$$

is the tip-to-tail sum of all those steps as you make your way along the path. Note that orientation matters; if you walk the path backwards, each arrow is flipped, and the whole integral changes sign. We define positively oriented contours with counter-clockwise paths (think of the right hand rule).

In this sense, complex integration does not have a geometric picture like "area under a curve" or "enclosed volume." Instead, the sum of vectors represents how the function f interacts with its surroundings as you move along the path. As we will now see, certain functions interact in such a perfectly balanced way that every closed loop gives zero, while singularities disturb this balance and leave a detectable imprint in the integral.

2.1 Upper semicircle example with $f(z) = z^2$



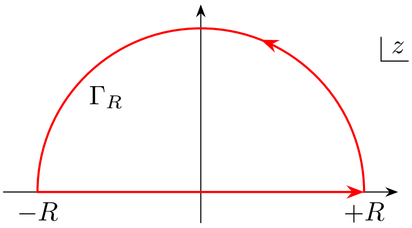
Let C_R be the upper semicircle of radius R centered at the origin, oriented counterclockwise:

$$z(\theta) = R\,e^{i\theta}, \qquad \theta \in [0, \pi], \qquad dz = iR\,e^{i\theta}\,d\theta.$$

Then

$$\int_{C_R} z^2\,dz = \int_0^{\pi} \left(R^2 e^{2i\theta}\right)\left(iRe^{i\theta}\right)d\theta = iR^3 \int_0^{\pi} e^{3i\theta}\,d\theta = iR^3 \left[\frac{e^{3i\theta}}{3i}\right]_0^{\pi} = \frac{R^3}{3}\left(e^{3i\pi} - 1\right) = -\frac{2}{3}\,R^3.$$

We notice that the evaluation of the integral depends on R, the radius of the contour we chose. If we instead close the contour by adding the diameter along the real axis from $-R$ to R, we get a closed loop Γ_R.



Since z^2 has the antiderivative $\frac{1}{3}z^3$, we compute the closed-loop integral over Γ_R by splitting it into its two parts:

$$\oint_{\Gamma_R} z^2\,dz = \int_{-R}^{R} z^2\,dz \;+\; \int_{C_R} z^2\,dz.$$

Along the diameter $[-R, R]$ the variable is real, $z = x$, so

$$\int_{-R}^{R} z^2\,dz = \int_{-R}^{R} x^2\,dx = \left[\tfrac{1}{3}x^3\right]_{-R}^{R} = \tfrac{2}{3}R^3.$$

Adding to the contribution from C_R gives

$$\oint_{\Gamma_R} z^2\,dz = \tfrac{2}{3}R^3 \;+\; \left(-\tfrac{2}{3}R^3\right) = 0.$$

In fact, we can tie this example to the more general result. Suppose f has a single-valued antiderivative F. If a contour C runs from z_1 to z_2 and is parametrized by $z = z(t)$, $a \le t \le b$, then by the chain rule

$$\frac{d}{dt}F(z(t)) \;=\; F'(z(t))\,z'(t) \;=\; f(z(t))\,z'(t).$$

Hence

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b \frac{d}{dt}F(z(t))\,dt = F(z(b)) - F(z(a)).$$

Since $z(a) = z_1$ and $z(b) = z_2$, this shows

$$\int_C f(z)\,dz \;=\; F(z_2) - F(z_1).$$

The crucial point is that the value of the integral depends only on the endpoints, not on the shape of the curve, provided the entire contour lies in the region where F is defined. In particular, if C is a closed contour with $z_1 = z_2$, the integral vanishes. Later we will see that this property breaks down if no single-valued antiderivative exists throughout the region; those are the situations where singularities or branch points come into play.

2.2 Singularities

The function

$$f(z) = \frac{1}{z^2},$$

is holomorphic everywhere except at the origin. An antiderivative is

$$F(z) = -\frac{1}{z},$$

which is defined and single–valued on the punctured plane $\mathbb{C} \setminus \{0\}$. Thus, for any closed contour C lying entirely in $\mathbb{C} \setminus \{0\}$, we have

$$\int_C \frac{dz}{z^2} = 0.$$

For instance, on the positively oriented circle

$$z = Re^{i\theta}, \qquad -\pi \leq \theta \leq \pi, \tag{2.1}$$

the integral vanishes.

There is a subtlety here worth emphasizing. Both $f(z) = 1/z^2$ and its antiderivative $F(z) = -1/z$ blow up at $z = 0$. How, then, can the "antiderivative trick" still work? The resolution is that we do not require F to exist everywhere in the plane - only on the domain where our contour lies. By deleting the origin, we obtain a domain $\mathbb{C} \setminus \{0\}$ on which F is perfectly single–valued and holomorphic, and therefore the integral of f around any closed loop in this domain vanishes.

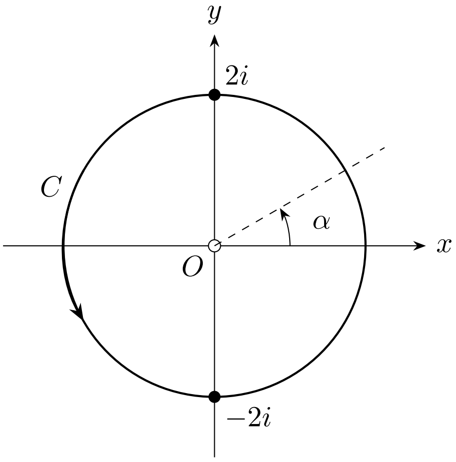
In general, when a function has singularities or branch points, we are free to *delete* those points (and, if necessary, curves extending to infinity, i.e. branch cuts) to obtain a new domain. The guiding principle is: 1) the domain must be open and connected, 2) the contour and its interior must lie entirely inside this domain, 3) on that domain the chosen antiderivative must be single–valued.

For isolated singularities like $1/z^2$, removing a point suffices. For multi–valued functions such as $\log z$, one must remove both the singularity and an additional cut to make the domain simply connected. Later, the residue theorem will tell us exactly how integrals "detect" these deleted singularities.

Note that the integral of the function $f(z) = 1/z$ around the same circle *cannot* be evaluated in the same way. Although the derivative of any branch $F(z)$ of $\log z$ is $1/z$, the function $F(z)$ is not differentiable, or even defined, along its branch cut. If a ray $\theta = \alpha$ from the origin is used to form the branch cut, then $F'(z)$ fails to exist at the point where that ray intersects the circle C. Thus C does not lie entirely in a domain throughout which $F'(z) = 1/z$, and we cannot directly use an antiderivative.

2.3 Example: 1/z

Here we illustrate that we can piece together *two different branches* of an antiderivative (two branches of $\log z$) to evaluate the integral of $f(z) = 1/z$ around the circle C.



Let C_1 denote the right half of the circle C above and take R = 2.

$$z = 2e^{i\theta}, \qquad -\tfrac{\pi}{2} \leq \theta \leq \tfrac{\pi}{2}. \tag{2.2}$$

The principal branch of the logarithm,

$$\operatorname{Log} z = \ln r + i\Theta, \qquad r > 0,\ -\pi < \Theta < \pi,$$

serves as an antiderivative of $1/z$ on C_1. Thus

$$\int_{C_1} \frac{dz}{z} = \int_{-2i}^{2i} \frac{dz}{z} = \Big[\operatorname{Log} z\Big]_{-2i}^{2i} = \operatorname{Log}(2i) - \operatorname{Log}(-2i).$$

Evaluating gives

$$\operatorname{Log}(2i) = \ln 2 + i\frac{\pi}{2}, \qquad \operatorname{Log}(-2i) = \ln 2 - i\frac{\pi}{2},$$

so

$$\int_{C_1} \frac{dz}{z} = \pi i.$$

Next, let C_2 denote the left half of the circle C:

$$z = 2e^{i\theta}, \qquad \tfrac{\pi}{2} \le \theta \le \tfrac{3\pi}{2}. \tag{2.3}$$

For this arc we instead consider the branch

$$\log z = \ln r + i\theta, \qquad r > 0,\ 0 < \theta < 2\pi,$$

which is also an antiderivative of $1/z$ on C_2. Then

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = \Big[\log z\Big]_{2i}^{-2i} = \log(-2i) - \log(2i).$$

Here

$$\log(-2i) = \ln 2 + i\frac{3\pi}{2}, \qquad \log(2i) = \ln 2 + i\frac{\pi}{2},$$

so

$$\int_{C_2} \frac{dz}{z} = \pi i.$$

Finally, the value of the integral of $1/z$ around the entire circle $C = C_1 + C_2$ is obtained by summing:

$$\int_{C} \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

This is the hallmark of a simple pole: the integral of $1/z$ around a positively oriented (if the contour is clockwise, we'd get an extra minus sign) circle enclosing the origin equals $2\pi i$.

2.4 Contour Deformation

So far, our evaluation of integrals relied on the existence of a single–valued antiderivative. But what if no such F exists globally? The key fact is that analyticity itself is enough to guarantee that closed integrals vanish and, more generally, that contour integrals are invariant under continuous deformations of the path that avoid singularities.

We can think of this in the context of electromagnetism. If we write $z = x + iy$, $dz = dx + i\,dy$, and $f = u + iv$ with u, v real-valued, then

$$(u + iv)(dx + i\,dy) = \underbrace{(u\,dx - v\,dy)}_{\text{real part}} + i\,\underbrace{(v\,dx + u\,dy)}_{\text{imag part}}.$$

So

$$\int_\gamma f\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy).$$

Each of these integrals has the same form as the circulation of a vector field, $\int_\gamma \mathbf{A}\cdot d\mathbf{r}$, with

$$\mathbf{A}_1 = (u,\,-v), \qquad \mathbf{A}_2 = (v,\,u).$$

Green's theorem relates the circulation around a closed curve to the curl inside the region it bounds. If D is a region in the plane and ∂D its positively oriented boundary curve, then for a vector field $\mathbf{A} = (P, Q)$ in the plane,

$$\oint_{\partial D} P\,dx + Q\,dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Therefore, for $f = u + iv$,

$$\oint \mathbf{A}_1\cdot d\mathbf{r} = \iint (\nabla\times\mathbf{A}_1)_z\,dA = \iint (-v_x - u_y)\,dA,$$

$$\oint \mathbf{A}_2\cdot d\mathbf{r} = \iint (\nabla\times\mathbf{A}_2)_z\,dA = \iint (u_x - v_y)\,dA.$$

If f is holomorphic, the Cauchy–Riemann equations $u_x = v_y,\ u_y = -v_x$ (the complex derivative must be the same whether we approach in the x- or y-direction) give

$$(\nabla\times\mathbf{A}_1)_z = 0, \qquad (\nabla\times\mathbf{A}_2)_z = 0,$$

so

$$\oint_C f\,dz = 0 \quad \text{for any closed } C \text{ in the region.}$$

Now suppose C_1 and C_2 are two contours that can be smoothly deformed into one another without crossing a singularity. As one contour is deformed into the other, the path they trace together carves out a region D: you follow C_1 forward, then C_2 in reverse, and the loop they form encloses exactly the swept-out area.

By Green's theorem,

$$\int_{C_1} f\,dz - \int_{C_2} f\,dz = \oint_{\partial D} f\,dz = \iint_D 0\,dA = 0,$$

so the two contour integrals are equal.

If the deformation crosses a singularity, the CR equations fail there (the "curl" has a source), and the closed integral need not vanish; this is the residue contribution we will see below.

2.5 Cauchy's Integral Formula

The deformation principle immediately leads to one of the most powerful results in complex analysis. Let f be analytic on a domain D, and let C be a positively oriented simple closed contour lying in D. Suppose a is a point in the interior of C. We want to understand

$$\oint_C \frac{f(z)}{z-a}\,dz.$$

The trick is to separate off the singular-looking part:

$$\frac{f(z)}{z-a} = \frac{f(z)-f(a)}{z-a} + \frac{f(a)}{z-a}.$$

The first term is analytic even at $z = a$

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a).$$

so its integral around C vanishes by deformation invariance. This leaves only

$$\oint_C \frac{f(z)}{z - a}\, dz = f(a) \oint_C \frac{dz}{z - a}.$$

Now shrink C down to a tiny circle centered at a. In polar coordinates $z = a + re^{i\theta}$, we have $dz = ire^{i\theta} d\theta$ and $z - a = re^{i\theta}$. Therefore

$$\oint_C \frac{dz}{z - a} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = \int_0^{2\pi} i\, d\theta = 2\pi i.$$

By deformation invariance, the same value holds for the original contour. We arrive at the *Cauchy integral formula*:

$$f(a) \;=\; \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a}\, dz.$$

This result is extraordinary: the value of an analytic function inside a region is completely determined by its values on the boundary. Moreover, differentiating under the integral sign yields

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}}\, dz,$$

so *all* derivatives of f are encoded by contour integrals.

The fact that holomorphic functions admit a power-series representation follows directly from Cauchy's integral formula . If f is holomorphic on U and C is a simple closed contour around z_0, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a}\, dz.$$

Expanding the kernel as a geometric series for a near z_0,

$$\frac{1}{z - a} = \frac{1}{(z - z_0) - (a - z_0)} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{a - z_0}{z - z_0} \right)^n,$$

and interchanging the sum with the integral gives

$$f(a) = \sum_{n=0}^{\infty} b_n (a - z_0)^n, \qquad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Thus the power series is not an extra assumption, but a consequence of holomorphicity itself. This rigidity is what makes complex analysis far stronger than real analysis: differentiability already forces a local power-series expansion.

3 Residue Theorem

If f is analytic around a point a, then by definition it can be expanded in a power series valid in some disk

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots, \qquad |z - a| < R.$$

Every term here has a perfectly good antiderivative, so for a closed contour C around a we obtain

$$\oint_C f(z)\,dz = 0.$$

If f has an *isolated singularity* at a, then f is instead analytic everywhere in a punctured neighborhood of a, on the region $0 < |z-a| < R$. Here f has no issues except exactly at a, where it blows up or is undefined. In this situation, the Taylor expansion centered at a is impossible: the series cannot converge all the way in to $z = a$, because the function itself fails to exist there. Yet f is still analytic on the *annulus* $0 < |z-a| < R$. Analyticity on an annulus is enough to guarantee a convergent *Laurent series*:

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots.$$

Here the integer $m \geq 0$ describes the “worst” power of $(z-a)$ that appears in the denominator.

Classification of singularities.

- If $m = 0$, there are no negative powers at all. The Laurent series reduces to an ordinary Taylor series. In this case the point a is not truly singular but *removable.*

  Consider

  $$f(z) = \frac{\sin z}{z}.$$

  At first glance the denominator suggests a singularity at $z = 0$. But expanding $\sin z$ into its Taylor series,

  $$\sin z = z - \tfrac{z^3}{3!} + \tfrac{z^5}{5!} - \cdots,$$

  and dividing by z gives

  $$\frac{\sin z}{z} = 1 - \tfrac{z^2}{3!} + \tfrac{z^4}{5!} - \cdots$$

  This is an ordinary power series with no negative powers of z. Therefore the singularity at $z = 0$ is only apparent. By defining $f(0) := 1$, we obtain an analytic function everywhere.

- If m is a positive integer, then a is a *pole of order m.* The Laurent series contains exactly m terms in the principal part (the negative powers), running from $(z-a)^{-m}$ up to $(z-a)^{-1}$.

- If no finite m suffices — in other words, the Laurent series requires infinitely many negative powers — then a is an *essential singularity.* Classic examples include $e^{1/z}$ or $\sin(1/z)$ at $z = 0$.

Thus the shape of the principal part of the Laurent series directly reveals the nature of the singularity. As our earlier examples with $1/z^n$ showed, all the terms with $n \neq 1$ possess single–valued antiderivatives on the punctured plane and therefore contribute nothing to a closed contour integral. The only exception is the $\frac{a_{-1}}{z-a}$ term,

$$f(z) = \cdots + \cancel{\tfrac{a_{-2}}{(z-a)^2}} + \boxed{\frac{a_{-1}}{z-a}} + \cancel{a_0} + \cancel{a_1(z-a)} + \cancel{a_2(z-a)^2} + \cdots$$

which behaves exactly like $1/z$ and contributes $2\pi i$. Thus

$$\oint_C f(z)\,dz = 2\pi i\, a_{-1}.$$

The coefficient a_{-1} is called the *residue* of f at a. Summing over several singularities inside C gives the general form of the *residue theorem*:

$$\oint_C f(z)\,dz \;=\; 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j),$$

where $z_1, \ldots, z_k$ are the isolated singularities of f inside C, and $\operatorname{Res}(f; z_j)$ denotes the residue of f at $z = z_j$.

3.1 Computing Residues

Since the residue of f at a is the coefficient of $\frac{1}{z-a}$, we want a way to isolate a_{-1}. For a pole of order m,

$$f(z) = \sum_{k=0}^{+\infty} a_k (z-z_0)^k + \frac{a_{-1}}{(z-z_0)} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-m}}{(z-z_0)^m}$$

if we multiply $f(z)$ with $(z-z_0)^m$, then we remove the negative powers of the expansion:

$$(z-z_0)^m f(z) = \sum_{k=0}^{+\infty} a_k (z-z_0)^{k+m} + a_{-1}(z-z_0)^{m-1} + a_{-2}(z-z_0)^{m-2} + \cdots + a_{-m}$$

If we take the derivative with respect to z, the constant a_{-m} drops out and the power of a_{-1} goes down by 1

$$\frac{d}{dz}\left[(z-z_0)^m f(z)\right] = \sum_{k=0}^{+\infty} a_k (k+m)(z-z_0)^{k+m-1} + a_{-1}(m-1)(z-z_0)^{m-2} + \cdots + a_{-m+1}$$

So we see that, in order to eliminate the $(z-z_0)^{m-1}$ in the a_{-1} term, we must apply the derivative $m-1$ times,

$$\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] = \cdots + a_{-1}(m-1)(m-2)\cdots 1 = \cdots + (m-1)!\, a_{-1}$$

Finally, taking the $z \to z_0$ limit leads to the general formula:

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{\,m-1}}{dz^{\,m-1}}\left[(z-z_0)^m f(z)\right]$$

Any term with power higher than $m-1$ (like $(z-z_0)^m$, $(z-z_0)^{m+1}$, $\ldots$) will still contain positive powers of $(z-z_0)$. $\Rightarrow$ When we take the limit $z \to z_0$, those terms vanish.

Any term with power lower than $m-1$ would have been killed earlier during differentiation. $\Rightarrow$ So they don't contribute.

The most common and easiest case is that of a simple pole: m = 1. Here f "blows up once" at a:

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0) f(z)$$

A very useful special case: if $f = \dfrac{g}{h}$ with g, h analytic near a, $h(a) = 0$, and $h'(a) \neq 0$ (so a is a simple zero of h), then

$$\operatorname{Res}\left(\frac{g}{h}; a\right) = \frac{g(a)}{h'(a)}$$

This is the go–to formula for rational functions with distinct linear factors in the denominator.

3.2 Choosing Contours

The residue theorem is always true for a closed contour. But when we try to evaluate a real integral $\int_{-\infty}^{\infty} f(x)\,dx$, the path is *not* closed. To apply the residue theorem we artificially close it so that

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\text{Arc}} f(z)\,dz.$$

We must then argue that the extra piece $\int_{\text{Arc}} f(z)\,dz$ goes away as $R \to \infty$. If it does, then the real line integral equals the sum of residues.

Along the large semicircle, $z = Re^{i\phi}$, the radius R is huge. So the only way the arc integral can shrink to zero is if the integrand itself becomes very small there. Two common mechanisms:

1. *Polynomial decay:* If $f(z)$ looks like $1/z^2$ or faster, then on the arc it behaves like $1/R^2$ and multiplying by the arc length($\sim \pi R$) tends to zero.

2. *Exponential damping:* If $f(z)$ carries a factor e^{+iz} and we close in the upper half plane (UHP), then $e^{+i(x+iy)} = e^{ix}e^{-y}$, so along the arc the e^{-y} kills the integral when we send $y \to \infty$. With e^{-iz} the damping happens in the lower half plane (LHP).

3.3 How Contour Integrals Stumped Feynman

Now let us return to the Feynman story from the start of the section. While we don't know the exact form of the integral that was presented to Feynman, here is the form that it could have taken.

Suppose $f(z)$ is an analytic function on the unit disk. Then, by Cauchy's integral formula,

$$\oint_\gamma \frac{f(z)}{z}\,dz = 2\pi i\, f(0),$$

where γ traces out the unit circle in a counterclockwise manner. Let $z = e^{i\phi}$. Then

$$\int_0^{2\pi} f(e^{i\phi})\,d\phi = 2\pi f(0).$$

Taking the real part of each side we find

$$\int_0^{2\pi} \operatorname{Re}\left(f(e^{i\phi})\right)d\phi = 2\pi \operatorname{Re}\left(f(0)\right)$$

(We could just as well take the imaginary part.) If we for example let

$$f(z) = \exp\left(\frac{2+z}{3+z}\right).$$

we find that

$$\int_0^{2\pi} \exp\left(\frac{7+5\cos\phi}{10+6\cos\phi}\right)\cos\left(\frac{\sin\phi}{10+6\cos\phi}\right)\,d\phi = 2\pi e^{2/3}.$$

Impressive as it looks, it is clear that far nastier integrals could be manufactured by a more devious choice of f.

4 Application to Quantum Field Theory

In quantum field theory, integrals from $-\infty$ to $+\infty$ naturally arise because fields and propagators are built from Fourier transforms. Spacetime variables like t and x are conjugate to energy and momentum, so when we write correlation functions or propagators in momentum space, we must integrate over the entire real axis of these variables. Restricting to only positive values would lose half the Fourier modes, breaking causality and Lorentz invariance.

However, these integrals often contain poles on the real axis, making them ill-defined if treated naively. This is where contour integration becomes powerful: by extending the real integral into the complex plane and using the residue theorem, we can evaluate such integrals in a controlled way.

4.1 Example

If we want to calculate the integral

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2 + A^2}, \quad A > 0$$

We can use contour integration

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2 + A^2} = \int_{-\infty}^{+\infty} \frac{dx}{(x - iA)(x + iA)} = 2\pi i \operatorname{Res}\left(\frac{1}{(x - iA)(x + iA)}, iA\right)$$

The poles are simple at $z = \pm iA$. For a simple pole $z = z_0$,

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Residue at $z = iA$

$$\operatorname{Res}(f, iA) = \lim_{z \to iA} \frac{z - iA}{(z - iA)(z + iA)} = \frac{1}{2iA}.$$

Residue at $z = -iA$

$$\operatorname{Res}(f, -iA) = \lim_{z \to -iA} \frac{z + iA}{(z - iA)(z + iA)} = \frac{1}{-2iA} = -\frac{1}{2iA}.$$

Equivalently, using the derivative rule for simple poles: $\operatorname{Res}(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$ for $f = g/h$;

Here $g(z) = 1$, $h(z) = z^2 + A^2$, so $h'(z) = 2z$. Therefore

$$\operatorname{Res}(f, iA) = \frac{1}{2iA}, \qquad \operatorname{Res}(f, -iA) = \frac{1}{-2iA}.$$

Now, closing the contour in the upper half-plane picks only the pole at $z = iA$, hence

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2 + A^2} = 2\pi i \ \operatorname{Res}(f, iA) = 2\pi i \left(\frac{1}{2iA}\right) = \frac{\pi}{A}.$$

If we instead closed in the lower half-plane, we would flip the sign since the contour was closed clock-wise

$$(-1)(2\pi i) \times \operatorname{Res}(f, -iA) = \frac{-2\pi i}{-2iA} = \frac{\pi}{A}$$

which agrees with our result for the UHP.

4.2 Example

$$I := \int_{-\infty}^{+\infty} \frac{\cos x}{x^2 + A^2}\, dx, \qquad A > 0.$$

Split into exponentials.

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}) \Rightarrow \quad I = \frac{1}{2}\left[\underbrace{\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + A^2}\, dx}_{=:I_1} + \underbrace{\int_{-\infty}^{+\infty} \frac{e^{-ix}}{x^2 + A^2}\, dx}_{=:I_2}\right].$$

The integrand $\dfrac{1}{z^2 + A^2}$ has simple poles at $z = \pm iA$.

Evaluate I_1 by closing in the upper half-plane

Only $z = iA$ lies inside. Because it is a simple pole,

$$\operatorname{Res}(f_1, iA) = \lim_{z \to iA} \frac{(z - iA)e^{iz}}{(z - iA)(z + iA)} = \frac{e^{i(iA)}}{2iA} = \frac{e^{-A}}{2iA}.$$

Apply residue theorem. With positive orientation,

$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2 + A^2}\, dx = 2\pi i \operatorname{Res}(f_1, iA) = 2\pi i \left(\frac{e^{-A}}{2iA}\right) = \frac{\pi}{A} e^{-A}.$$

Evaluate I_2 by closing in the lower half-plane

Write $z = x + iy$; on the LHP we have $y < 0$. Then

$$e^{-iz} = e^{-i(x+iy)} = e^{-ix+y},$$

so $|e^{-iz}| = e^y \to 0$ as $y \to -\infty$. Again the arc integral $\to 0$.

Residues inside the LHP. Only $z = -iA$ lies inside. For the simple pole,

$$\operatorname{Res}(f_2, -iA) = \frac{e^{-i(-iA)}}{(2z)\big|_{z=-iA}} = \frac{e^{-A}}{-2iA}.$$

Orientation sign. The LHP contour is clockwise, so

$$I_2 = \int_{-\infty}^{+\infty} \frac{e^{-ix}}{x^2 + A^2}\, dx = -\,2\pi i \operatorname{Res}(f_2, -iA) = -2\pi i \left(\frac{e^{-A}}{-2iA}\right) = \frac{\pi}{A} e^{-A}.$$

Combine I_1 and I_2.

$$I = \frac{1}{2}(I_1 + I_2) = \frac{1}{2}\left(\frac{\pi}{A}e^{-A} + \frac{\pi}{A}e^{-A}\right) = \boxed{\frac{\pi}{A}e^{-A}}.$$